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CORRECTIONS OF FORMULAS IN "AN ELEMENTARY DERIVATION OF MOMENTS OF HAWKES PROCESSES"

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In the course of some research on Hawkes process parameter estimation with missingness, we needed to calculate the moments of the event count over a fixed time window. We found many useful results in [1], which provides elegant and easily understood tools and a theoretical framework for deriving these moments. During our explorations of their many valuable results, we discovered a couple minor inconsistencies between formulas provided for some expected values and the simulated values of those quantities in our work. To address these issues, we carefully revisited the calculations and discovered some small omissions in two results from [1]. We corrected these errors and compiled our findings into this correction note. In the following sections, we present the corrected formulas along with step-by-step derivations and simulation results that validate these corrections.

1. Correction 1: expression of $\mathbb{E}[\lambda^2(t)]$

On p.114 of [1], for $\alpha \neq \beta$, expression of $\mathbb{E}[\lambda^2(t)]$ is

$$\mathbb{E}[\lambda^2(t)] = \left[\frac{\Delta_{11}}{\alpha-\beta} + \nu^2 - \frac{\Delta_{12}}{\alpha-\beta}\right] e^{-2(\alpha-\beta)t} \tag{1}$$

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where

$$\Delta_{11} = (\alpha^2 + 2\beta\nu)\nu \left[\frac{\alpha e^{3(\alpha-\beta)t}}{3(\alpha-\beta)} - \frac{\beta e^{2(\alpha-\beta)t}}{2(\alpha-\beta)}\right]$$
$$\Delta_{12} = (\alpha^2 + 2\beta\nu)\nu \left[\frac{\alpha}{3(\alpha-\beta)} - \frac{\beta}{2(\alpha-\beta)}\right]$$

The corrected formula is

$$\mathbb{E}[\lambda^2(t)] = \frac{\nu\left[(\alpha^2+2\beta\nu)(\beta-2\alpha e^{(\alpha-\beta)t}) + \alpha^2(2\alpha-\beta+2\nu)e^{2(\alpha-\beta)t}\right]}{2(\alpha-\beta)^2} \tag{2}$$

2. Correction 2: Eq.(13)

In Eq.(13) of [1], for $\alpha \neq \beta$, expression of $\mathbb{E}[N(t)\lambda(t)]$ is

$$\mathbb{E}[N(t)\lambda(t)] = e^{(\alpha-\beta)t}\int_0^t \beta\nu\mathbb{E}[N(u)] + \alpha\mathbb{E}[\lambda(u)] + \mathbb{E}[\lambda^2(u)]du \tag{3}$$

The corrected formula is

$$\mathbb{E}[N(t)\lambda(t)] = e^{(\alpha-\beta)t}\int_0^t e^{-(\alpha-\beta)u}\left\{\beta\nu\mathbb{E}[N(u)] + \alpha\mathbb{E}[\lambda(u)] + \mathbb{E}[\lambda^2(u)]\right\}du \tag{4}$$

3. Derivation of correction 1

Solving ODE Eq.(12) in [1], with initial condition $\mathbb{E}[\lambda^2(0)] = \nu^2$,

$$\frac{d}{dt}\mathbb{E}[\lambda^2(t)] = (2\beta\nu+\alpha^2)\mathbb{E}[\lambda(t)] + (2\alpha-2\beta)\mathbb{E}[\lambda^2(t)]$$

Assume $\alpha \neq \beta$, in Eq.(10) of [1] shows that

$$\mathbb{E}[\lambda(t)] = -\frac{\beta\nu}{\alpha-\beta} + \frac{\alpha\nu}{\alpha-\beta}e^{(\alpha-\beta)t}$$

Let $y(t) = \mathbb{E}[\lambda^2(t)]$. Substituting $\mathbb{E}[\lambda(t)]$ into the ODE, and simplify the terms:

$$\frac{dy}{dt} = \frac{(2\beta\nu+\alpha^2)(-\beta\nu)}{\alpha-\beta} + \frac{(2\beta\nu+\alpha^2)(\alpha\nu)}{\alpha-\beta}e^{(\alpha-\beta)t} + (2\alpha-2\beta)y(t)$$

Reorganizing the equation:

$$\frac{d}{dt}\left(e^{-(2\alpha-2\beta)t}y(t)\right) = \frac{(2\beta\nu+\alpha^2)(\alpha\nu)}{\alpha-\beta}e^{-(\alpha-\beta)t} + \frac{(2\beta\nu+\alpha^2)(-\beta\nu)}{\alpha-\beta}e^{-(2\alpha-2\beta)t}$$

Let $C_1 = \frac{(2\beta\nu+\alpha^2)(\alpha\nu)}{\alpha-\beta}$ and $C_2 = \frac{(2\beta\nu+\alpha^2)(-\beta\nu)}{\alpha-\beta}$, the general solution for $y(t)$ is:

$$y(t) = -\frac{C_1}{\alpha-\beta}e^{(\alpha-\beta)t} - \frac{C_2}{2\alpha-2\beta} + C_3 e^{(2\alpha-2\beta)t}$$

Determining the Constant C_3 by condition $y(0) = \nu^2$, the final solution for $\mathbb{E}[\lambda^2(t)]$ is:

$$\mathbb{E}[\lambda^2(t)] = \frac{\nu\left[(\alpha^2+2\beta\nu)(\beta-2\alpha e^{(\alpha-\beta)t}) + \alpha^2(2\alpha-\beta+2\nu)e^{2(\alpha-\beta)t}\right]}{2(\alpha-\beta)^2}$$

3.1. Comparison of the two formulas

Formula from [1] can be reformulated as following:

$$\begin{aligned}\mathbb{E}[\lambda^2(t)] = & -\frac{(\alpha^2+2\beta\nu)\beta\nu}{2(\alpha-\beta)^2} + \frac{(\alpha^2+2\beta\nu)\alpha\nu}{3(\alpha-\beta)^2}e^{(\alpha-\beta)t} \\ & + \left[\nu^2 - \frac{(\alpha^2+2\beta\nu)(2\alpha-3\beta)\nu}{6(\alpha-\beta)^2}\right]e^{-2(\alpha-\beta)t}\end{aligned}$$

The corrected formula:

$$\begin{aligned}\mathbb{E}[\lambda^2(t)] &= \frac{\nu\left[(\alpha^2+2\beta\nu)(\beta-2\alpha e^{(\alpha-\beta)t}) + \alpha^2(2\alpha-\beta+2\nu)e^{2(\alpha-\beta)t}\right]}{2(\alpha-\beta)^2} \\ &= \frac{(\alpha^2+2\beta\nu)\beta\nu}{2(\alpha-\beta)^2} - \frac{(\alpha^2+2\beta\nu)\alpha\nu}{(\alpha-\beta)^2}e^{(\alpha-\beta)t} + \frac{\alpha^2(2\alpha-\beta+2\nu)\nu}{2(\alpha-\beta)^2}e^{2(\alpha-\beta)t}\end{aligned}$$

Notice that for $\alpha < \beta$ (stationary condition for Hawkes process with exponential kernel), the term $e^{-2(\alpha-\beta)t}$ in expression of [1] will dominate the expected value when t gets large.

3.2. Check the formula by Monte-Carlo simulation

Simulation setup:

- number of simulated datasets to get MC estimator: $n_{\text{sim}} = 1,000,000$;
- baseline intensity: low, medium, high $\nu = 0.5, 2.0, 5.0$;
- combinations of α, β: $(\alpha, \beta) = \{(0.5, 1.0), (1.0, 1.5), (1.5, 2.0)\}$;
- simulation time horizon: $T = 10$.

Figure 1 shows the simulation comparison results. See appendix A for simulation code, and appendix B for complete simulation tables.

4. Derivation of correction 2

ODE Eq.(8) in [1] is

$$\frac{d}{dt}\mathbb{E}[N(t)\lambda(t)] = \beta\nu\mathbb{E}[N(t)] + \alpha\mathbb{E}[\lambda(t)] + (\alpha-\beta)\mathbb{E}[N(t)\lambda(t)] + \mathbb{E}[\lambda^2(t)]$$

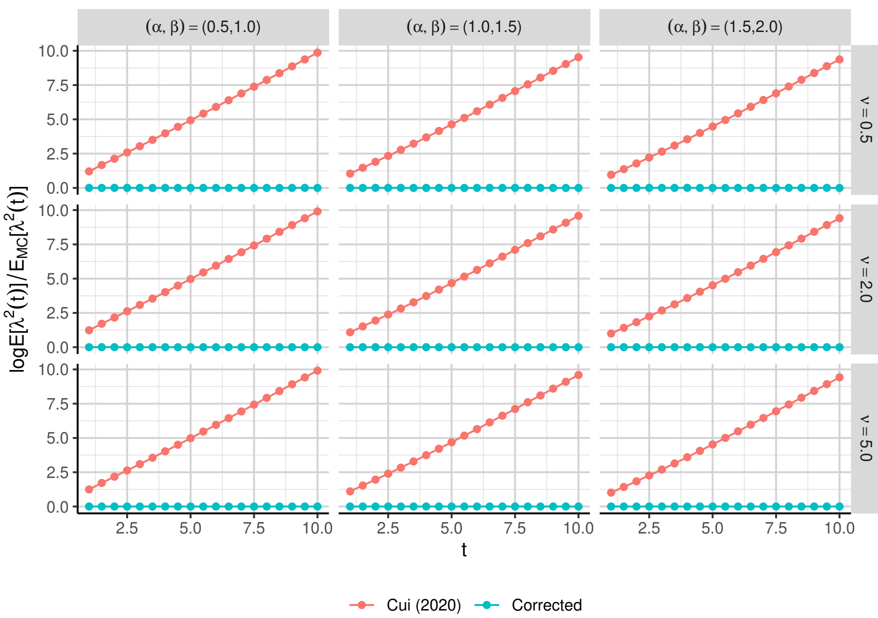

FIGURE 1: Plot of the logarithm of the ratio $\mathbb{E}[\lambda^2(t)]/\mathbb{E}_{\mathrm{MC}}[\lambda^2(t)]$, where $\mathbb{E}_{\mathrm{MC}}[\lambda^2(t)]$ is a Monte Carlo estimator based on 1,000,000 simulated Hawkes process realizations. The red line corresponds to the formula from [1], and the blue line represents the corrected formula. When the formula closely matches the MC estimator, the logarithmic ratio is expected to be close to 0.

With initial condition $\mathbb{E}[N(0)\lambda(0)] = 0$, for $\alpha \neq \beta$. denote $g(t) = \beta\nu\mathbb{E}[N(t)] + \alpha\mathbb{E}[\lambda(t)] + \mathbb{E}[\lambda^2(t)]$ and $f(t) = \mathbb{E}[N(t)\lambda(t)]$, then

$$
\begin{aligned}
& f'(t) = (\alpha-\beta)f(t) + g(t) \\
\Longleftrightarrow\ & e^{-(\alpha-\beta)t}f'(t) - (\alpha-\beta)e^{-(\alpha-\beta)t}f(t) = e^{-(\alpha-\beta)t}g(t) \\
\Longleftrightarrow\ & f(t) = e^{(\alpha-\beta)t}\int_0^t e^{-(\alpha-\beta)u}g(u)du
\end{aligned}
$$

which leads to the corrected formula.

Appendix A. Simulation R code

```
El2_Cui2020 <- function(t, alpha, beta, nu) {
  # Define Delta_11 and Delta_12
```

```
  Delta_11 <- (alpha^2 + 2 * beta * nu) * nu * ((alpha * exp(3 *
      (alpha - beta) * t)) / (3 * (alpha - beta)) - (beta * exp(2
      * (alpha - beta) * t)) / (2 * (alpha - beta)))
  Delta_12 <- (alpha^2 + 2 * beta * nu) * nu * ((alpha) / (3 *
      (alpha - beta)) - (beta) / (2 * (alpha - beta)))
  # Compute E[lambda^2(t)]
  E_lambda2 <- ((Delta_11 / (alpha - beta)) + nu^2 - (Delta_12 /
      (alpha - beta))) * exp(-2 * (alpha - beta) * t)
  return(E_lambda2)
}

El2 <- function(t, alpha, beta, nu) {
  # Compute the Delta term
  term1 <- (alpha^2 + 2 * beta * nu) * (beta - 2 * alpha *
      exp((alpha - beta) * t))
  term2 <- alpha^2 * (2 * alpha - beta + 2 * nu) * exp(2 * (alpha
      - beta) * t)
  # Compute E[lambda^2(t)]
  E_lambda2 <- (nu * (term1 + term2)) / (2 * (alpha - beta)^2)
  return(E_lambda2)
}

El2_sim <- function(t, alpha, beta, nu, horizon, nsim){
  # get intensity sequence from event sequence
  lambda_func <- function(t, nu, alpha, beta, hvec) {
    nu + sum(alpha * exp(-beta * (t - hvec[hvec < t])))
  }
  lamb_vec <- rep(0, nsim)
  for (i in 1:nsim){
    h <- simulateHawkes(nu, alpha, beta, horizon)
    lamb_vec[i] <- lambda_func(t, nu, alpha, beta, h[[1]])
  }
  return(mean(lamb_vec**2))
}
```

Appendix B. Complete simulation results

The complete simulation data for figure 1 are listed in this section.

	$(\nu,\alpha,\beta)=(0.5,0.5,1.0)$			$(\nu,\alpha,\beta)=(0.5,1.0,1.5)$			$(\nu,\alpha,\beta)=(0.5,1.5,2.0)$		
t	simulation	Cui (2020)	corrected	simulation	Cui (2020)	corrected	simulation	Cui (2020)	corrected
1.0	0.5838	1.9475	0.5838	1.2689	3.6035	1.2692	2.4295	6.31	2.4222
1.5	0.7159	3.802	0.7153	1.7262	7.4946	1.7229	3.4887	13.6752	3.4814
2.0	0.8227	6.9081	0.824	2.1065	14.1043	2.1136	4.4141	26.3142	4.4185
2.5	0.9108	12.0671	0.9124	2.4344	25.1533	2.4406	5.2062	47.5389	5.2164
3.0	0.9842	20.6023	0.9835	2.7068	43.4881	2.709	5.8596	82.8332	5.8791
3.5	1.0429	34.6975	1.0403	2.9334	73.809	2.9264	6.4163	141.2581	6.4203
4.0	1.0849	57.9544	1.0854	3.1026	123.8712	3.1008	6.8306	237.7669	6.8569
4.5	1.1214	96.3125	1.121	3.2433	206.4656	3.2397	7.2173	397.025	7.2062
5.0	1.1495	159.5651	1.1491	3.3464	342.6842	3.3497	7.479	659.7081	7.4837
5.5	1.1707	263.8596	1.1711	3.4491	567.3044	3.4365	7.733	1092.8854	7.7033
6.0	1.1883	435.8186	1.1884	3.5015	937.6668	3.5048	7.8718	1807.1412	7.8764
6.5	1.2013	719.3363	1.2019	3.5535	1548.3118	3.5584	8.0407	2984.8021	8.0124
7.0	1.2136	1186.7818	1.2125	3.5967	2555.111	3.6004	8.1033	4926.4776	8.1191
7.5	1.2204	1957.4724	1.2207	3.6202	4215.0548	3.6332	8.2164	8127.7908	8.2026
8.0	1.2281	3228.1288	1.2272	3.6472	6951.8492	3.6589	8.2412	13405.8888	8.268
8.5	1.2363	5323.0889	1.2322	3.679	11464.0677	3.679	8.3081	22108.0204	8.319
9.0	1.235	8777.0956	1.2361	3.697	18903.4643	3.6946	8.3697	36455.4249	8.3589
9.5	1.2364	14471.791	1.2392	3.7198	31168.9604	3.7069	8.3687	60110.3075	8.39
10.0	1.2419	23860.7574	1.2416	3.7177	51391.3481	3.7164	8.4501	99110.6247	8.4143

TABLE 1: $\mathbb{E}[\lambda^2(t)]$, low baseline intensity: $\nu = 0.5$

References

[1] CUI, L., HAWKES, A. AND YI, H. (2020). An elementary derivation of moments of Hawkes processes. *Adv. Appl. Probab.* **52,** 102–137.

	$(\nu,\alpha,\beta)=(2.0,0.5,1.0)$			$(\nu,\alpha,\beta)=(2.0,1.0,1.5)$			$(\nu,\alpha,\beta)=(2.0,1.5,2.0)$		
t	simulation	Cui (2020)	corrected	simulation	Cui (2020)	corrected	simulation	Cui (2020)	corrected
1.0	8.2647	28.1173	8.1605	14.5456	43.6216	14.6561	24.2077	65.1904	23.9513
1.5	9.8131	54.3960	9.8623	19.6019	89.3170	19.5638	33.3788	139.0430	33.9398
2.0	11.1962	98.3835	11.2874	23.8154	166.8346	23.8348	43.1017	265.5906	42.8409
2.5	12.4025	171.4218	12.4578	27.7445	296.3363	27.4333	49.3659	477.9589	50.4537
3.0	13.3201	292.2426	13.4059	30.0147	511.1698	30.4009	56.4426	830.9975	56.7954
3.5	14.1742	491.7550	14.1666	32.6290	866.3995	32.8122	62.8329	1415.3201	61.9847
4.0	14.7923	820.9385	14.7726	35.2826	1452.8757	34.7509	66.5204	2380.4655	66.1780
4.5	15.2412	1363.8599	15.2526	36.0098	2420.4357	36.2976	69.1740	3973.0923	69.5358
5.0	15.7946	2259.1336	15.6315	37.2677	4016.1586	37.5245	70.9204	6599.9576	72.2068
5.5	15.8227	3735.3052	15.9296	38.7363	6647.4395	38.4936	75.0370	10931.7580	74.3208
6.0	16.2156	6169.1903	16.1635	39.0808	10985.9830	39.2565	76.3268	18074.3370	75.9877
6.5	16.4520	10182.0581	16.3469	39.5037	18139.2618	39.8557	76.9601	29850.9632	77.2984
7.0	16.4057	16798.2129	16.4903	40.3921	29933.2033	40.3254	77.4802	49267.7302	78.3268
7.5	16.3865	27706.4503	16.6024	40.7459	49378.2650	40.6930	78.7690	81280.8729	79.1322
8.0	16.6186	45691.1263	16.6900	40.5462	81437.8602	40.9804	81.3169	134061.8604	79.7623
8.5	16.8053	75342.8697	16.7583	41.2344	134295.2812	41.2049	80.8251	221083.1826	80.2547
9.0	16.9241	124230.3498	16.8116	41.2354	221442.5014	41.3801	78.9046	364557.2322	80.6391
9.5	16.8105	204832.1937	16.8532	42.1848	365124.0282	41.5169	81.3882	601106.0621	80.9392
10.0	16.9294	337722.1804	16.8856	41.5623	602014.8575	41.6235	83.2435	991109.2370	81.1733

TABLE 2: $\mathbb{E}[\lambda^2(t)]$, medium baseline intensity: $\nu = 2.0$

	$(\nu,\alpha,\beta)=(5.0,0.5,1.0)$			$(\nu,\alpha,\beta)=(5.0,1.0,1.5)$			$(\nu,\alpha,\beta)=(5.0,1.5,2.0)$		
t	simulation	Cui (2020)	corrected	simulation	Cui (2020)	corrected	simulation	Cui (2020)	corrected
1.0	49.5244	171.9294	49.5276	84.5889	255.0912	84.5375	131.1669	362.7278	131.1909
1.5	59.6597	331.9302	59.6607	112.3285	519.9865	112.2714	184.8433	769.3196	184.92
2.0	68.1637	599.7145	68.1757	136.6197	969.1744	136.4888	233.0627	1465.6446	232.936
2.5	75.2186	1044.3217	75.1854	157.0226	1719.4554	156.9374	273.8839	2633.9146	274.0741
3.0	80.8994	1779.7737	80.8738	173.818	2964.0106	173.826	308.5594	4575.8168	308.3829
3.5	85.4005	2994.2128	85.4431	187.7093	5021.8158	187.5631	336.1636	7789.7391	336.4797
4.0	89.0875	4997.9513	89.086	198.7118	8419.1429	198.6159	359.5735	13098.1542	359.1968
4.5	92.0093	8302.7001	91.9743	207.2187	14023.955	207.4387	377.2432	21857.6913	377.3955
5.0	94.257	13752.1994	94.2547	214.503	23267.5059	214.4403	392.5871	36305.5208	391.8759
5.5	96.1016	22737.5979	96.0496	219.8054	38509.7082	219.9722	403.6284	60130.4772	403.3396
6.0	97.4267	37552.5551	97.4588	224.4525	63641.5363	224.3284	412.3002	99414.704	412.3806
6.5	98.5491	61978.7105	98.5632	227.6523	105078.2278	227.7501	419.9211	164186.1814	419.4903
7.0	99.4117	102250.9599	99.4276	230.5772	173396.8045	230.4326	425.4268	270978.4256	425.0689
7.5	100.1413	168648.9289	100.1033	232.5246	286035.8908	232.5324	429.2857	447050.7303	429.4388
8.0	100.5791	278120.8712	100.631	234.2975	471746.9683	234.1742	432.6993	737346.177	432.8573
8.5	101.0319	458609.7458	101.043	235.5297	777933.2546	235.4568	435.8226	1215963.462	435.529
9.0	101.3369	756185.7128	101.3644	236.5637	1282749.474	236.4581	437.7434	2005070.744	437.6153
9.5	101.6059	1246805.633	101.6151	236.9182	2115051.004	237.2393	439.2467	3306089.315	439.2435
10.0	101.8246	2055701.204	101.8105	237.7819	3487284.47	237.8486	440.1727	5451106.783	440.5137

TABLE 3: $\mathbb{E}[\lambda^2(t)]$, high baseline intensity: $\nu = 5.0$